Centre of Gravity of Surface and Solid of Revolution.

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THE formulæ derived in this paper are for the most general case of the revolution of any curve, plane or tortuous, about any axis, through any angle.

Let $\rho = \phi(t) = \phi$ (t being omitted for brevity) be the equation of any curve, and let ϵ be a unit vector in any direction; then

$$\rho = \epsilon^{\frac{\theta}{\pi}} \phi \epsilon^{-\frac{\theta}{\pi}} \tag{1}$$

will be the equation of a surface of revolution formed by revolving $\phi(t)$ about a line through the origin in the direction ϵ.

As the origin may be moved to any point by introducing a constant vector into $\phi(t)$, and as any direction may be chosen for ϵ, this equation is perfectly general.

1st. The *surface* element is $TVD_\theta\rho D_t\rho \,.\, d\theta dt$.

$$D_\theta\rho = \epsilon^{\frac{2\theta}{\pi}} V\epsilon\phi, \quad \text{and} \quad D_t\rho = \epsilon^{\frac{\theta}{\pi}} \phi' \epsilon^{-\frac{\theta}{\pi}};$$

hence

$$TVD_\theta\rho D_t\rho = TV\epsilon^{\frac{2\theta}{\pi}} V\epsilon\phi\epsilon^{\frac{\theta}{\pi}}\phi'\epsilon^{-\frac{\theta}{\pi}} = \sqrt{\phi'^2 V^2\epsilon\phi - S^2\epsilon\phi\phi'} = TV\phi' V\epsilon\phi,$$

as may be shown by expanding and reducing. Therefore the surface is

$$S = \int_{t_1}^{t_2}\int_{\theta_1}^{\theta_2} d\theta dt TV\phi' V\epsilon\phi = (\theta_2 - \theta_1)\int_{t_1}^{t_2} dt TV\phi' V\epsilon\phi. \tag{2}$$

We have then for the centre of gravity, if the density be uniform,

$$\bar{\rho} = \iint \epsilon^{\frac{\theta}{\pi}}\phi\epsilon^{-\frac{\theta}{\pi}} TV\phi' V\epsilon\phi \,.\, d\theta dt \div \iint TV\phi' V\epsilon\phi \,.\, d\theta dt; \tag{3}$$

or, integrating for θ from θ_1 to θ_2,

$$\bar{\rho} = \int \left[\theta\epsilon^{-1}S\epsilon\phi - \epsilon^{\frac{2\theta}{\pi}} V\epsilon\phi\right]_{\theta_1}^{\theta_2} TV\phi' V\epsilon\phi \,.\, dt \div (\theta_2 - \theta_1)\int TV\phi' V\epsilon\phi \,.\, dt. \tag{4}$$

If the integration be from 0 to 2π, equation (4) becomes

$$\rho = \epsilon^{-1} \int S\epsilon\phi T V\phi' V\epsilon\phi \,.\, dt \div \int TV\phi' V\epsilon\phi \,.\, dt. \tag{5}$$

If the generating curve lie in a plane through the origin and the vector ϵ about which it is revolved, we have $S\epsilon\phi\phi' = 0$, and therefore $TV\phi' V\epsilon\phi = T\phi' V\epsilon\phi$, so that in that case this latter expression may be substituted for the former.

2d. *Volume.* Writing $\sigma = u\rho = u\epsilon^{\frac{\theta}{\pi}}\phi\epsilon^{-\frac{\theta}{\pi}}$, we have

$$V = -\int\int\int SD_u\sigma D_\theta\sigma D_t\sigma \,.\, dud\theta dt = -\int\int\int u^2 S\epsilon^{\frac{\theta}{\pi}}\phi\epsilon^{-\frac{\theta}{\pi}} VD_\theta\rho D_t\rho \,.\, dud\theta dt,$$

which on expansion and reduction becomes,

$$V = \int\int\int u^2 S \,.\, \phi\phi' V\epsilon\phi \,.\, dud\theta dt; \tag{6}$$

and for the centre of gravity

$$\bar{\rho} = \int\int\int u^3 \epsilon^{\frac{\theta}{\pi}}\phi\epsilon^{-\frac{\theta}{\pi}} S \,.\, \phi\phi' V\epsilon\phi \,.\, dud\theta dt \div \int\int\int u^2 S \,.\, \phi\phi' V\epsilon\phi \,.\, dud\theta dt. \tag{7}$$

If we integrate for u from 0 to 1, and for θ from θ_1 to θ_2, we have

$$\rho = \tfrac{1}{4}\int [\theta\epsilon^{-1}S\epsilon\phi - \epsilon^{\frac{2\theta}{\pi}} V\epsilon\phi]_{\theta_1}^{\theta_2} S \,.\, \phi\phi' V\epsilon\phi \,.\, dt \div \tfrac{1}{3}(\theta_2 - \theta_1)\int S \,.\, \phi\phi' V\epsilon\phi \,.\, dt. \tag{8}$$

If $\theta_1 = 0$ and $\theta_2 = 2\pi$, equation (10) becomes

$$\bar{\rho} = \tfrac{1}{4}\epsilon^{-1}\int S\epsilon\phi S \,.\, \phi\phi' V\epsilon\phi \,.\, dt \div \tfrac{1}{3}\int S \,.\, \phi\phi' V\epsilon\phi \,.\, dt. \tag{9}$$

Formulæ (7), (8), and (9) are unchanged when the generating curve lies in a plane through the axis, though in that case $TV\phi\phi' V\epsilon\phi$ may be substituted for $S \,.\, \phi\phi' V\epsilon\phi$.

As an example of these formulæ let us take the hyperboloid of one sheet generated by the revolution of a right line about an axis which it does not intersect.

Let $\phi(t) = \alpha + t\beta$; then the hyperboloid will be

$$\rho = \epsilon^{\frac{\theta}{\pi}}(\alpha + \beta t)\epsilon^{-\frac{\theta}{\pi}}. \tag{10}$$

Let $S\alpha\epsilon = S\alpha\beta = 0$, and $T\beta = 1$; then

$$TV\phi' V\epsilon\phi = \sqrt{\phi'^2 V^2\epsilon\phi - S^2\epsilon\phi\phi'} = \sqrt{-\alpha^2 - S^2\epsilon\alpha\beta - t^2 V^2\epsilon\beta} = TV\epsilon\beta\sqrt{a^2 + t^2},$$

if we write
$$\frac{T^2\alpha - S^2\epsilon\alpha\beta}{T^2 V\epsilon\beta} = a^2; \tag{11}$$

$$\therefore \rho = \int [\theta t\epsilon^{-1}S\epsilon\beta - \epsilon^{\frac{2\theta}{\pi}} V\epsilon(\alpha + \beta t)]_{\theta_1}^{\theta_2} dt\sqrt{a^2 + t^2} \div (\theta_2 - \theta_1)\int dt\sqrt{a^2 + t^2}$$

$$= \left[\epsilon^{-1}S\epsilon\beta - \frac{(\epsilon^{\frac{2\theta_2}{\pi}} - \epsilon^{\frac{2\theta_1}{\pi}})V\epsilon\beta}{\theta_2 - \theta_1}\right]\frac{\int tdt\sqrt{a^2+t^2}}{\int dt\sqrt{a^2+t^2}} - \frac{(\epsilon^{\frac{2\theta_2}{\pi}} - \epsilon^{\frac{2\theta_1}{\pi}})V\epsilon\alpha}{\theta_2 - \theta_1}. \tag{12}$$

The integration of this equation will give the centre of gravity of a strip of the surface of the hyperboloid lying between two positions of the generatrix at an angular distance apart of $\theta_2 - \theta_1$.

For the solid we have

$$S\phi\phi' V\epsilon\phi = S(\alpha + \beta t)\beta V\epsilon(\alpha + \beta t) = a^2 S\epsilon\beta;$$

so that

$$\bar{\rho} = \tfrac{1}{4}\int_0^{t_1}[\theta t\epsilon^{-1}S\epsilon\beta - \epsilon^{\frac{2\theta}{\pi}}V\epsilon(\alpha + \beta t)]_{\theta_1}^{\theta_2}dt \div \tfrac{1}{3}(\theta_2 - \theta_1)t_1,$$

$$= \frac{3}{4}\left[\frac{1}{2}t_1\epsilon^{-1}S\epsilon\beta - \frac{1}{\theta_2 - \theta_1}(\epsilon^{\frac{2\theta_2}{\pi}} - \epsilon^{\frac{2\theta_1}{\pi}})V\epsilon(\alpha + \frac{1}{2}t_1\beta\right]. \tag{13}$$

This gives the centre of gravity of the solid bounded by the plane $S\epsilon\rho = 0$, the hyperboloid, a cone with its vertex at the origin cutting the hyperboloid in the circle $\rho = \epsilon^{\frac{\theta}{\pi}}\phi(t_1)\epsilon^{-\frac{\theta}{\pi}}$, and two positions of the plane through the origin and the generating line given by θ_1 and θ_2.

It will be noticed on comparison that the formulæ derived in this article do not agree with equations (23), (25), and (26) of Mr. Stringham's article in No. 3, Vol. II., of this Journal. Those equations appear to be incorrect for the reason that, though the integration has already been performed with regard to one or more variables, ρ is still taken as the arm of the element, which *should* be the vector from the origin to the centre of gravity of the element. It would follow from each of these equations, since they are independent of ϕ, that the barycentric vector is independent of the angular distance through which the generating curve has been revolved, which is certainly not the fact.